

Math Lab 14 MS 2

Time Constants of Exponential Growth Processes

For best results, print this document front-to-back and place it in a three-ring binder.
Corresponding teacher and student pages will appear on each opening.

TEACHING PATH - MATH SKILLS LAB - CLASS M

RESOURCE MATERIALS

Student Text: Math Skills Lab

CLASS GOALS

1. Distinguish between shapes of curves for exponential growth and exponential decay processes.
2. Learn how to identify the $1/e$ -time constant for an exponential growth process from the curve plotted for that process.

CLASS ACTIVITIES

1. Take five or ten minutes to go through the Student Exercises. Make sure that students understand the correct answers.
2. Complete as many activities as time permits. Students already should have read the discussion material and looked at the examples for each activity before coming to this class. (How much you accomplish will depend on the math skills your students already have.) Summarize the explanatory material for the Math Activity, "Time Constants of Exponential Growth Processes." Then have students complete the Practice Exercises given at the end of the Math Activity.
3. Supervise student progress. Help students obtain the correct answers.
4. Before the class ends, tell your students to read Lab 14*3, "Time Constants for RC Circuits," before coming to the lab class.

Math Skills Laboratory

Lab 14^{MS} 2

MATH ACTIVITY

Activity: Time Constants of Exponential Growth Processes

MATH SKILLS LAB OBJECTIVES

When you complete these activities, you should be able to:

1. Discuss the similarities and differences between exponential growth processes and exponential decay processes.
2. Analyze an exponential growth process and find the time constant.

LEARNING PATH

1. Read the Math Skills Lab. Give particular attention to the Math Skills Lab Objectives.
2. Work the problems.

ACTIVITY

Time Constants of Exponential Growth Processes

Many of the processes that decay in an exponential manner are reversible. That is, they can be made to go in the opposite direction. For example, you can discharge a capacitor. That's a **decay** process. But you can also charge it. That's the reverse process, a **growth** process.

When this reversing takes place, the growth process is mathematically described by the equation:

$$Q = Q_{\max} (1 - e^{-t/\tau})$$

where: Q = quantity increasing in the growth process
 Q_{\max} = final value toward which "Q" is increasing
 t = time during which the change takes place
 τ = 1/e time constant

A graph that shows "Q" versus time (t) for exponential growth and exponential decay is shown in Figure 1. The time (t) along the horizontal axis is marked off in units of the 1/e time constant (τ)— τ , 2τ , 3τ , etc.

NOTE: The two curves in Figure 1 are important and contain much information. Make a transparency of Figure 1, project it on a screen or wall, and point out each of the 5 similarities and each of the 3 differences itemized below Figure 1.

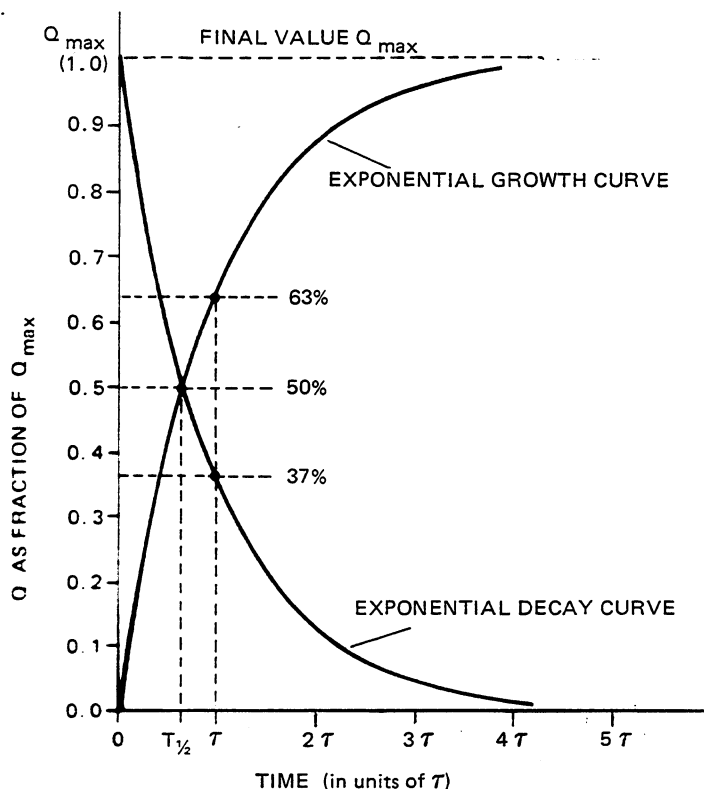


Fig. 1 Exponential growth and decay curves.

If you compare the curve for exponential growth with the curve for exponential decay, you'll see many similarities and differences. Some of the similarities between the two curves are:

- Both curves approach, but never reach, a limiting value.
- Both curves show a fast initial rate of change, but this rate gets slower as time increases.
- Both curves give the same value for $T_{1/2}$.
- Both curves have the same value for τ .
- After five $1/e$ time constants (5τ), both curves reach over 99% of their limiting value.

Some of the differences between the curves are:

- For the decay curve, $Q = 0.368 Q_{\max}$ (about 37% Q_{\max}) after one time constant (τ).
- For the growth curve, $Q = 0.632 Q_{\max} = (1 - 0.368 Q_{\max})$ after one time constant (about 63% Q_{\max}).
- The decay curve starts at a maximum value and approaches a minimum value. The growth curve starts at a minimum value and approaches a maximum.

Let's examine the curve of a typical growth process. The circuit shown in Figure 2 can be used to measure the voltage across a capacitor during the charging process. Consider the charging of a $7.5\text{-}\mu\text{F}$ capacitor through a $250\text{-}\Omega$ resistor with a 100-V DC source. The voltage across this capacitor (as a function of time) is given by the equation:

$$V = V_{\max}(1 - e^{-t/\tau})$$

If $V_{\max} = 100$ volts, this equation gives the data shown in Table 1.

The data in Table 1 is plotted on the graph shown in Figure 3. Note how the voltage changes rapidly at the beginning, near $t = 1$ msec.

By contrast, note how the voltage changes much more slowly as the charging process continues—say at $t = 7$ msec. Note also that the voltage approaches $V_{\max} = 100$ volts very slowly.

(This page is intentionally blank.)

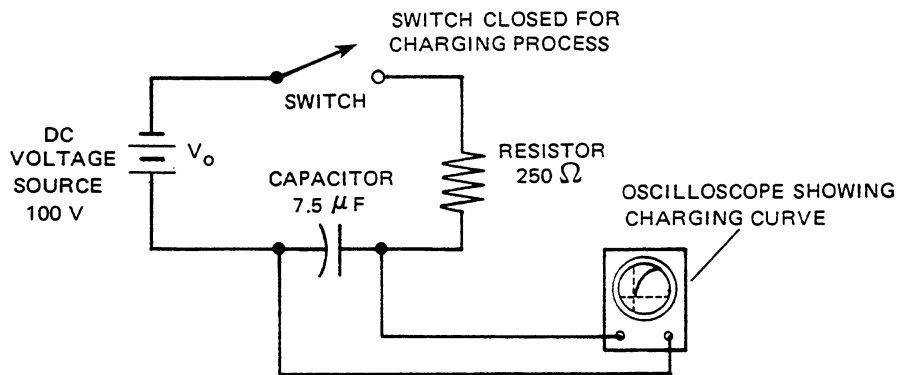


Fig. 2 Charging a capacitor.

TABLE 1. VOLTAGE ACROSS CAPACITOR (IN FIGURE 2) DURING THE CHARGING PROCESS

V	t
0 V	0 msec
41.3 V	1 msec
65.6 V	2 msec
79.8 V	3 msec
88.2 V	4 msec
95.9 V	6 msec
98.6 V	8 msec
99.5 V	10 msec

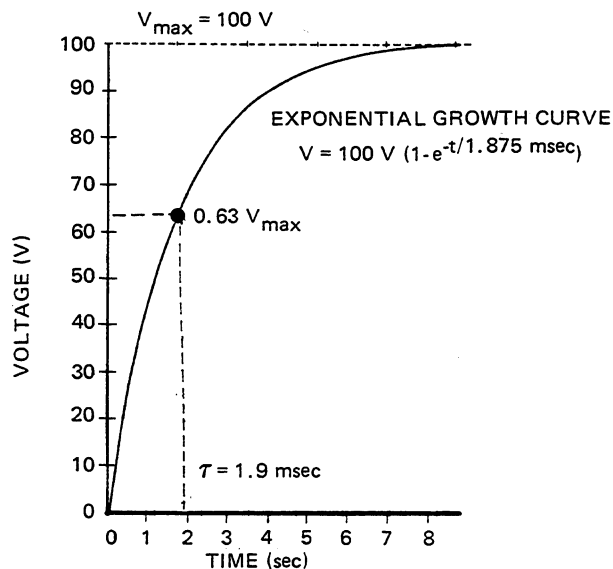


Fig. 3 Voltage versus time curve.

The $1/e$ time constant (τ) for the capacitor charging process is determined from the graph. A horizontal line is drawn at $V = 0.63 V_{max}$ (63 volts) over to the curve. An intersection point is located on the curve. A vertical line is drawn from the intersection point to the time axis.

The vertical line crosses the time axis at approximately 1.9 msec. This value is the best you can "read" off the graph. (The correct value—obtained from the equation $\tau = R \times C$ —is $\tau = 1.875 \text{ msec}$. So the approximate value of $\tau = 1.9 \text{ msec}$ is quite acceptable.)

SOLUTIONS TO PRACTICE EXERCISES

Problem 1

- a. The time $t = 2$ msec is equal to the time constant $\tau = 2$ msec. Hence you know that in 2 msec, the voltage across the capacitor will reach about 63% of the final value--100 volts. Sixty-three percent of 100 V-- $(0.63 \times 100 \text{ V})$ --is 63 volts.

Answer: After 2 msec, voltage on the capacitor is 63 volts.

- b. Solve this problem as outlined in Activity 3 of Lab 14MS1.

$$V = V_f (1 - e^{-t/\tau}) \quad \text{where: } V_f = 100 \text{ V}$$

$$t = 3 \text{ msec}$$

$$\tau = 2 \text{ msec}$$

$$V = 100 \text{ V} \left(1 - e^{-\frac{3 \text{ msec}}{2 \text{ msec}}} \right)$$

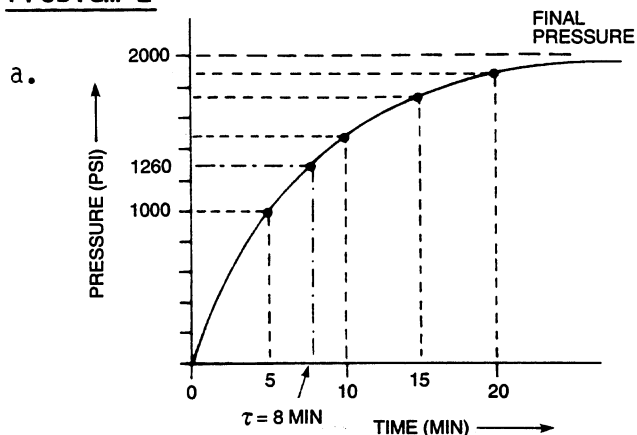
Use electronic calculator to evaluate $e^{-3/2} = e^{-1.5}$ to get 0.223.

Substitute 0.223 back into the equation and simplify:

$$V = 100 \text{ V} (1 - 0.223) = 100 \text{ V} (0.777) = 77.7 \text{ V}$$

$$\underline{V = 77.7 \text{ V}} \quad (\text{Voltage on capacitor after 3 msec})$$

Problem 2



In time $t = \tau$, pressure changes 63% of 2000 psi.

$(2000 \times 0.63 = 1260 \text{ psi})$. Locate 1260 psi on pressure axis and find corresponding time. That is the $1/e$ -time constant. From graph, τ equals about 8 min. (Plotting graph on fine grid paper will give a more accurate result.)

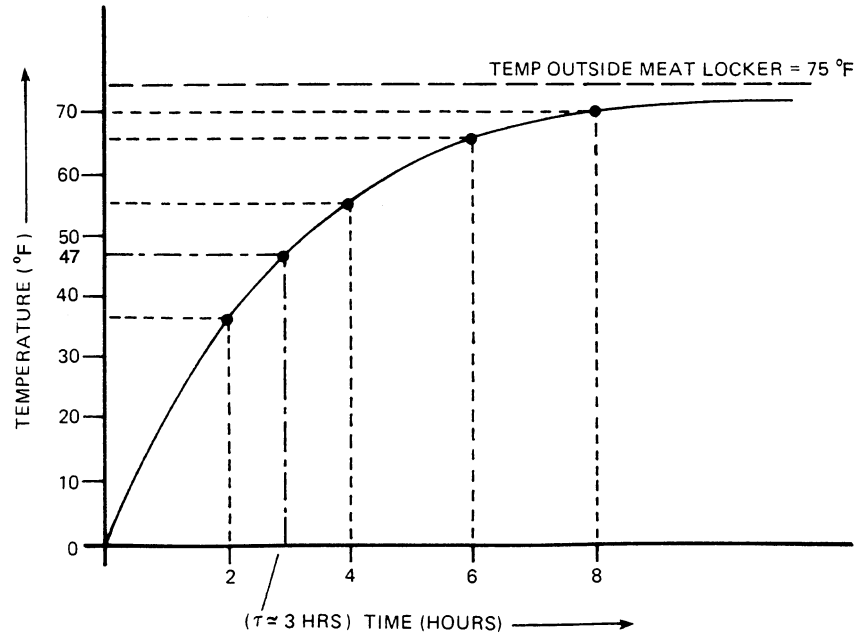
- b. The tank will be 99% charged in a time 5τ or about 40 min.

Answer to Problem 3 follows on Page T-68c.

SOLUTIONS TO PRACTICE EXERCISES, Continued

Problem 3

a.



To find the time constant τ , locate 63% of 75°F on "temperature axis"--then locate corresponding time on "time axis." Sixty-three percent of 75 is equal to 47.25 or about 47°F. This corresponds to a time of about 3 hours. Thus, 1/e-time constant $\tau \approx 3$ hours.

- b. In 3 hours the cold meat warms to 47°F. If the meat can remain above 47°F for no more than 8 hours, the company has a total of 11 hours (counting from time of power interruption) to restore electrical power—or 3 hours from the time the meat reaches the 70° temperature point.

PRACTICE EXERCISES

Complete the following problems.

Problem 1: Given: Demitri is repairing an RC circuit that filters the output of a 100-V dc power supply. The RC circuit has a $1/e$ time constant of 2 msec.

- Find:
- The voltage on the capacitor 2 msec after the power supply is turned ON.
 - The voltage on the capacitor 3 msec after the power supply is turned ON.

Solution: (**Hint:** Solve the equation, $V = 100(1 - e^{-t/2 \text{ msec}})$, for V at 2 msec and 3 msec. Refer to Activity 3 of Math Lab 14MS1.)

Problem 2: Given: Amy is charging air tanks for the Scuba Club. The charging source is a storage tank that's maintained at 2,000 psi by a compressor. Amy has found that the air tanks being charged have a pressure of 1,000 psi after 5 minutes, 1,500 psi after 10 minutes, 1,750 psi after 15 minutes and 1,875 psi after 20 minutes.

- Find:
- The $1/e$ time constant for the charging process.
 - The time required for the charging process to be essentially "completed."

Solution: (**Hint:** Plot the pressure versus time on a graph and determine the time constant.)

Problem 3: Given: Austin is working in a meat-packing plant. The power to the meat-locker refrigeration system is interrupted during a thunderstorm. The temperature outside the meat locker is 75°F, and the temperature in the meat locker is maintained at 0°F. Austin notices that the air temperature in the locker is 38°F after the power has been off for 2 hours, 56°F after 4 hours, 66°F after 6 hours and 70°F after 8 hours.

- Find:
- The $1/e$ time constant for the warming process.
 - The time available for the meat-packing company to restore the power if the meat can be kept at a temperature greater than 47°F for **no more** than 8 hours before the power (and refrigeration system) is turned on again.

Solution: (**Hint:** Plot the temperature versus time for the warming process. Determine the $1/e$ time constant $[\tau]$ from the graph.)